International Journal of Biomathematics
Vol. 4, No. 1 (March 2011) 109–118

DOI: 10.1142/S1793524511001167

DISPERSION STUDY OF AXIALLY SYMMETRIC WAVES IN CYLINDRICAL BONE FILLED WITH MARROW

P. MALLA REDDY* and B. SANDHYA RANI

Department of Mathematics, Kakatiya University
Warangal, Andhra Pradesh, 506009, India
**mperati@yahoo.com*

M. TAJUDDIN

Department of Mathematics, Osmania University
Hyderabad, Andhra Pradesh, 500007, India

Received 2 July 2009
Revised 21 May 2010

Bone filled with marrow is modeled as poroelastic bore filled with a fluid in the frame work of Biot's theory in wave propagation phenomena. Axially symmetric unattentuated waves are considered. Frequency equations are derived in the cases of permeable boundary and impermeable boundary, and are found to be dispersive. The existing data of bony materials has been exploited. Phase velocity versus ratio of wavelength to bone diameter curves are plotted. From this model and analytical results, some conclusions are drawn.

Keywords: Poroelasticity; bone; marrow; wave propagation; permeable boundary; impermeable boundary; frequency equation; phase velocity.

1. Introduction

Osteoarthritis (OA) affects more than half of the population above the age of 65 [1, 2] and has significant negative impact on the individuals quality of life [3]. Although OA has been considered to be primarily articular cartilage (interface between two bones) disorder, this disease is accompanied by change in trabecular bone (compact part of the bone) and marrow [1, 2]. Interactions between the bone and marrow may be useful in detecting the OA progression. Since direct investigation of bone is a destructive one, a study of wave propagation in bony materials is feasible, an alternative non-destructive way. It is clear that most of current knowledge of bone is based on waves in it.

Adult human skeleton consists of 206 distinct bones. Each consists of two parts: one is cancellous (spongy type, more porous) and other is compact as far as their mechanical behavior is concerned. They differ in mass densities. In either case, bones tend to exhibit a linearly elastic behavior over small strains, Hooke's law thus

applies. It was suggested that the theory of two phase poroelastic materials rather than elastic materials is to be applied in the sense that osseous tissue is considered as perfectly elastic and fluid substance filling the pores as a compressible fluid [4]. In general, bony materials are non-homogeneous, anisotropic and time dependent, however, from the Mathematics point of view, they are assumed as homogeneous isotropic materials so that Biot's theory of poroelastic materials can be used [4].

From the Mathematics perspective of poroelastic materials, plane strain vibrations of thick-walled hollow poroelastic cylinder is studied [5] following Biot's theory of wave propagation in poroelastic materials [6]. Wave propagation in cylindrical empty poroelastic bore is investigated [7]. Also, cylindrical stress waves in poroelastic flat slabs has been studied [8]. With regard to the actual problem considered here, we facilitate the Mathematical side of the problem by modelling the bone filled with marrow as a cylindrical poroelastic bore of infinite extent containing a fluid in the frame work of Biot's theory [6]. Waves of axial symmetry and unattentuated are considered here following the paper [9]. We are dealing essentially with the interaction between marrow and bone in wave propagation phenomena along the axis of bone. Nowinski and Davis studied longitudinal waves in cylindrical bone elements and computed some basic parameters of bone [4]. But in this paper marrow is not taken into consideration. Hence the study of bone filled with marrow is warranted. Because of axial symmetry, there are two stress components σ_{rr}, σ_{rz} and fluid pressure s. Three boundary conditions on the curved surface pertaining to the absence of external loads and permeability of the surface give frequency equations. Phase velocity is computed as a function of ratio of wavelength to diameter of bone both for permeable boundary and impermeable boundary.

The rest of the paper is organized as follows. Empty cylindrical bone is considered in Sec. 2. In Sec. 3, first behavior of waves in marrow fluid independently from the solid bone is investigated; next the bone containing marrow is examined. The non-dimensionalisation as well as numerical results are discussed in Sec. 4. Finally, concluding remarks are given in Sec. 5.

2. Empty Cylindrical Bone

Consider cylindrical polar coordinate system (r, θ, z) with z-axis along the axis of cylindrical poroelastic bone of infinite extent. Let the bone be homogeneous and isotropic. The equations of motion of a poroelastic solid given by Biot [6] in presence of dissipation (b) which in terms of displacement vectors are:

$$\begin{aligned} N\nabla^2\overrightarrow{u} + \nabla[(A+N)e + Q\epsilon] &= \frac{\partial^2}{\partial t^2}(\rho_{11}\overrightarrow{u} + \rho_{12}\overrightarrow{U}) + b\frac{\partial}{\partial t}(\overrightarrow{u} - \overrightarrow{U}), \\ \nabla[Qe + R\epsilon] &= \frac{\partial^2}{\partial t^2}(\rho_{12}\overrightarrow{u} + \rho_{22}\overrightarrow{U}) - b\frac{\partial}{\partial t}(\overrightarrow{u} - \overrightarrow{U}). \end{aligned} \tag{2.1}$$

In (2.1), P (= A + 2N), N, Q, R are all poroelastic constants, ∇^2 is the Laplacian operator, ρ_{ij} are mass coefficients, e and ϵ are solid dilatation and fluid dilatation,

respectively. The solid displacement components $(u, 0, w)$ which are functions of r, z and t (here the problem is plane strain and is independent of θ) which can readily be evaluated from the field equations of Biot [6] in matrix notation are

$$\begin{bmatrix} \dfrac{u}{-\cos(kz)} \\ \dfrac{w}{-\sin(kz)} \end{bmatrix} = \begin{bmatrix} pK_1(pr) & qK_2(qr) & kK_{2(dr)} \\ kK_0(pr) & kK_0(qr) & dK_0(dr) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ A_1 \end{bmatrix} e^{i\omega t}, \tag{2.2}$$

where ω is the frequency of wave, k is the wavenumber, C_1, C_2, A_1 are all constants, $K_n(x)$ is the modified Bessel function of second kind of order n, and

$$p = k(1-\xi_1^2)^{\frac{1}{2}}, \quad q = k(1-\xi_2^2)^{\frac{1}{2}}, \quad d = k(1-\xi_3^2)^{\frac{1}{2}}, \quad \xi_i = \frac{\omega}{kV_i} \quad (i = 1, 2, 3). \tag{2.3}$$

In Eq. (2.3), $V_i\,(i = 1, 2, 3)$ are dilatational wave velocities of first and second kind and shear wave velocity, respectively. Using these displacements into stress-displacement relations [6], the relevant stresses (σ_{ij}) and liquid pressure (s) are obtained as follows:

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{rz} \\ s \end{bmatrix} = \begin{bmatrix} A_{11}\cos(kz) & A_{12}\cos(kz) & A_{13}\cos(kz) \\ A_{21}\sin(kz) & A_{22}\sin(kz) & A_{23}\sin(kz) \\ A_{31}\cos(kz) & A_{32}\cos(kz) & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ A_1 \end{bmatrix} e^{i\omega t}, \tag{2.4}$$

where

$$\begin{aligned}
A_{11} &= \tfrac{2Np}{r}K_1(pr) + \{2Np^2 + (P-2N+Q\delta_1^2)(p^2-k^2) + (Q+R\delta_1^2)(p^2-k^2)\}K_0(pr), \\
A_{13} &= \tfrac{2kN}{r}K_1(dr) + 2NkdK_0(dr), \\
A_{21} &= 2NkpK_1(pr), \\
A_{23} &= N(d^2-k^2)K_1(dr), \\
A_{31} &= (Q+R\delta_1^2)(p^2-k^2)K_0(pr), \\
A_{33} &= 0, \\
\delta_i^2 &= \tfrac{(PR-Q^2)V_i^{-2}-(Rm_{11}-Qm_{12})}{Rm_{12}+Qm_{22}}, \ i = 1, 2 \\
m_{11} &= \rho_{11} - ibp^{-1}, m_{12} = \rho_{12} + ibp^{-1}, m_{22} = \rho_{22} - ibp^{-1}
\end{aligned}$$

A_{12}, A_{22}, A_{32} are similar expressions as A_{11}, A_{21}, A_{31} with p and δ_1^2 replaced by q and δ_2^2, respectively.

(2.5)

The boundary conditions to be satisfied on the curved surface $r = a$ to be stress free are

$$\sigma_{rr} + s = 0, \quad \sigma_{rz} = 0, \quad s = 0 \quad \text{(for permeable boundary)},$$

and

$$\sigma_{rr} + s = 0, \quad \sigma_{rz} = 0, \quad \frac{\partial s}{\partial r} = 0 \quad \text{(for impermeable boundary)}. \tag{2.6}$$

Equations (2.4) and (2.6) together give a system of three homogeneous equations for the constants C_1, C_2 and A_1 each for a permeable boundary and an impermeable

boundary. In order to obtain a nontrivial solution of this system the coefficients matrix must be singular. Accordingly, the frequency equation in the case of permeable boundary is given by

$$|a_{ij}| = 0, \quad i, j = 1, 2, 3, \tag{2.7}$$

where a_{ij} are same as A_{ij} in (2.4) but are to be evaluated at $r = a$.

In the case of an impermeable boundary, the frequency equation is given by

$$|b_{ij}| = 0, \quad i, j = 1, 2, 3, \tag{2.8}$$

where

$$b_{ij} = a_{ij} \quad i = 1, 2 \quad j = 1, 2, 3,$$

$$\begin{aligned} b_{31} &= (Q + R\delta_1^2)p(p^2 - k^2)K_1(pa), \\ b_{32} &= (Q + R\delta_2^2)q(q^2 - k^2)K_1(qa) \quad \text{and} \quad b_{33} = 0. \end{aligned}$$

These frequency Eqs. (2.7) and (2.8) will be examined numerically in Sec. 4.

3. Bone Filled With Marrow

In this section, first, the behavior of waves in marrow is studied independently from the solid part of bone. For axially symmetric waves, the displacement potential function ϕ satisfies the wave equation

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = \frac{1}{c_f^2}\frac{\partial^2\phi}{\partial t^2}, \tag{3.1}$$

where c_f is the velocity of the sound in marrow. The solution ϕ is given by

$$\begin{aligned} \phi &= J_0[rk(\zeta^2 - 1)^{\frac{1}{2}}]\cos(kz)e^{i\omega t} \quad \text{for } \zeta > 1, \\ \phi &= I_0[rk(1 - \zeta^2)^{\frac{1}{2}}]\cos(kz)e^{i\omega t} \quad \text{for } \zeta < 1. \end{aligned} \tag{3.2}$$

In (3.2), J_0 is the Bessel function of zero order of the first kind and I_0 is the modified zero order Bessel function of the first kind, and $\zeta = \frac{w}{kc_f}$ is the phase velocity along z-direction. The solution pertaining to $\zeta > 1$ corresponds to conical waves which are reflected at the boundary while other one corresponds to Stonely waves that occur at the fluid-solid interface. Thus the fluid pressure P_f $(= -\rho_f \frac{\partial^2\phi}{\partial t^2})$ and the radial displacement D_r $(= \frac{\partial\phi}{\partial r})$ can be computed. On the boundary $r = a$, one can have

$$\begin{aligned} \frac{P_f}{D_r} &= -\frac{\omega^2\rho_f J_0[ka(\zeta^2 - 1)]^{\frac{1}{2}}}{k(\zeta^2 - 1)^{\frac{1}{2}} J_1[ka(\zeta^2 - 1)]^{\frac{1}{2}}} \quad \text{for } \zeta > 1 \quad \text{and} \\ \frac{P_f}{D_r} &= -\frac{\omega^2\rho_f I_0[ka(1 - \zeta^2)]^{\frac{1}{2}}}{k(\zeta^2 - 1)^{\frac{1}{2}} I_1[ka(1 - \zeta^2)]^{\frac{1}{2}}} \quad \text{for } \zeta < 1. \end{aligned} \tag{3.3}$$

If the fluid is contained in a rigid wall, that is $D_r = 0$, then the former case gives

$$\frac{\zeta^2}{1} + \frac{(\frac{\lambda}{\Pi D})^2}{\alpha_i^2} = 1, \tag{3.4}$$

which is an ellipse.

The later case gives

$$\frac{\zeta^2}{1} - \frac{(\frac{\lambda}{\Pi D})^2}{\beta_i^2} = 1, \tag{3.5}$$

which is hyperbola.

In Eqs. (3.4) and (3.5), α_i, β_i are the roots of Bessel functions of order one of first kind and second kind, respectively, and $D = 2a$, diameter of the bone.

Now we consider bone filled with marrow. The boundary conditions to be satisfied on the curved surface $r = a$ to be stress free are

$$\frac{\sigma_{rr} + s}{u} = \frac{-P_f}{D_r}, \quad \sigma_{rz} = 0, \ \ s = 0 \quad \text{(for permeable boundary)},$$

and

$$\frac{\sigma_{rr} + s}{u} = \frac{-P_f}{D_r}, \quad \sigma_{rz} = 0, \ \ \frac{\partial s}{\partial r} = 0 \quad \text{(for impermeable boundary)}. \tag{3.6}$$

The first boundary condition in each case represents a condition at the fluid-solid interface, while the other conditions remain the same as in earlier section. Above boundary conditions also can be put in the following form

$$\sigma_{rr} + s = -P_f, \quad \sigma_{rz} = 0, \quad s = 0, u = D_r$$

and

$$\sigma_{rr} + s = -P_f, \quad \sigma_{rz} = 0, \frac{\partial s}{\partial r} = 0, \quad u = D_r.$$

However, former form is parsimonious, in the sense, it gives set of three homogeneous equations in three arbitrary constants C_1, C_2 and A_1, whereas, later case gives four equations in four arbitrary constants. For a nontrivial solutions of these constants the matrix of coefficients must be singular. Accordingly, the frequency equations in the case of permeable boundary is:

$$|c_{ij}| = 0, \quad i, j = 1, 2, 3 \quad \text{for } \zeta > 1, \tag{3.7}$$

where

$$\begin{aligned}
c_{11} &= A_{11} k(\zeta^2 - 1)^{\frac{1}{2}} J_1(ka(\zeta^2 - 1)^{\frac{1}{2}}) + pK_1(pa)\rho_f \omega^2 J_0(ka(\zeta^2 - 1)^{\frac{1}{2}}), \\
c_{12} &= A_{12} k(\zeta^2 - 1)^{\frac{1}{2}} J_1(ka(\zeta^2 - 1)^{\frac{1}{2}}) + qK_1(qa)\rho_f \omega^2 J_0(ka(\zeta^2 - 1)^{\frac{1}{2}}), \\
c_{13} &= A_{13} k(\zeta^2 - 1)^{\frac{1}{2}} J_1(ka(\zeta^2 - 1)^{\frac{1}{2}}) + kK_1(da)\rho_f \omega^2 J_0(ka(\zeta^2 - 1)^{\frac{1}{2}}), \\
c_{ij} &= A_{ij} \quad i = 2, 3, \quad j = 1, 2, 3.
\end{aligned}$$

For the case $\zeta < 1$, the frequency equation is given by

$$|d_{ij}| = 0, \quad i, j = 1, 2, 3, \tag{3.8}$$

where d_{ij} are same as c_{ij} with $(\zeta^2 - 1)^{\frac{1}{2}}$ is replaced by $(1 - \zeta^2)^{\frac{1}{2}}$.

Similarly, the frequency equations can be obtained for the case of impermeable boundary. All these frequency equations will be examined numerically in Sec. 4.

4. Numerical Results

In absence of dissipation, the frequency equations obtained in last section are analysed by introducing the following non-dimensional parameters:

$$\begin{aligned}
&a_1 = \frac{P}{H}, \quad a_2 = \frac{Q}{H}, \quad a_3 = \frac{R}{H}, \quad a_4 = \frac{N}{H},\\
&d_1 = \frac{\rho_{11}}{\rho}, \quad d_2 = \frac{\rho_{12}}{\rho}, \quad d_3 = \frac{\rho_{22}}{\rho},\\
&\tilde{x} = \left(\frac{V_0}{V_1}\right)^2, \quad \tilde{y} = \left(\frac{V_0}{V_2}\right)^2, \quad \tilde{z} = \left(\frac{V_0}{V_3}\right)^2,\\
&\rho = \rho_{11} + 2\rho_{12} + \rho_{22}, \quad H = P + 2Q + R, \quad V_0^2 = \frac{H}{\rho},\\
&m = \frac{c}{c_0} \quad \text{non-dimensional phase velocity}, \quad c = \frac{\omega}{k} \quad \text{and} \quad c_0^2 = \frac{N}{\rho}.
\end{aligned} \tag{4.1}$$

Using these non-dimensional parameters into the frequency Eqs. (2.7) and (2.8) of empty bone, one obtains an implicit relation between phase velocity (m) and the ratio of wavelength to diameter $\left(\frac{\lambda}{D}\right)$. Phase velocity is computed against this ratio both for permeable boundary and impermeable boundary. These equations yield infinitely many values for phase velocity, of which, first three values are presented in Fig. 1. The values of bone poroelastic parameters A, N, Q, R and its mass coefficients ρ_{ij} are computed as in the paper [4]. The values of Young's modulus and Poisson ratio are taken to be 3×10^6 and 0.28, respectively as suggested in the paper [4]. Velocities of dilatational waves and shear wave (V_1, V_2 and V_3) are computed using Biot's theory [6]. From Fig. 1, it is clear that the values of permeable boundary and the values of impermeable boundary are almost equal except in the neighborhood of $\frac{\lambda}{D} = 1.5$. In this neighborhood, the values of impermeable boundary are greater than that of permeable boundary. This deviation decreases from the value 1 to value 3. The first values of phase velocity for permeable boundary are almost same. In order to investigate the effectiveness of poroelastic constants A, N, Q, R on phase velocity, we make sensitivity analysis by changing the Poisson ratio. It is clear from the paper [4] that poroelastic constants A, N, Q, R are functions of Poisson ratio. Therefore, by changing the Poisson ratio values from 0.26 to 0.3 in step of 0.1, the pertaining values of A, N, Q, R, V_1, V_2 and V_3 are computed. Then for each set of these values, the phase velocity is computed against $\left(\frac{\lambda}{D}\right)$ both for

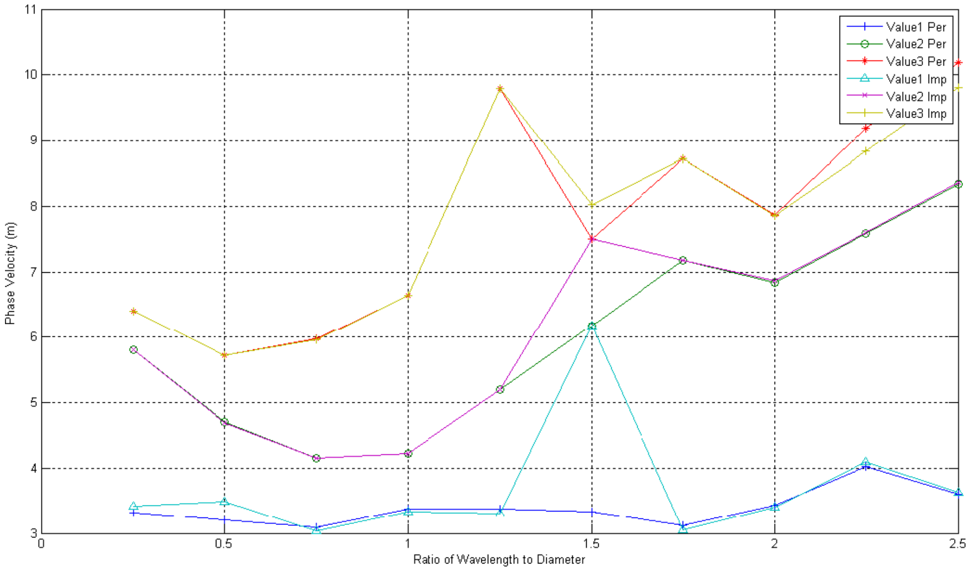

Fig. 1. Variation of phase velocity with wavelength/diameter in the case of empty bone for both permeable boundary and impermeable boundary.

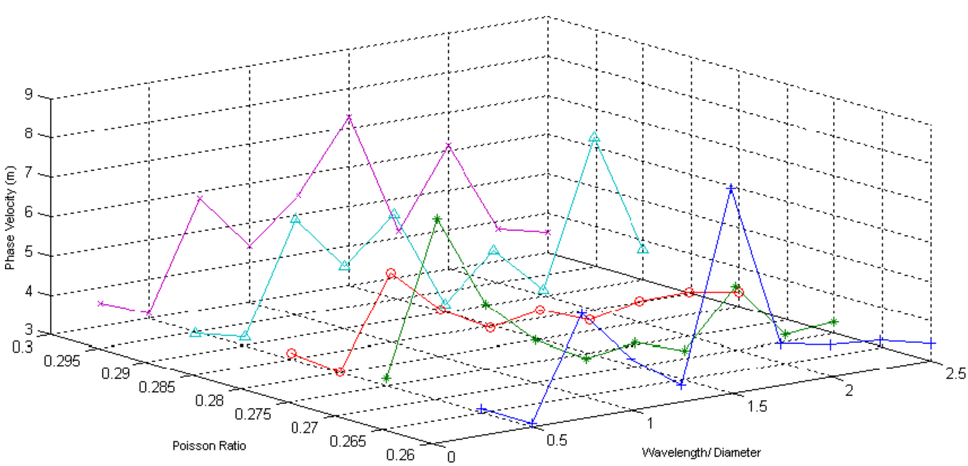

Fig. 2. Variation of phase velocity with wavelength/diameter at various poisson ratio in the case of empty bone for permeable boundary.

permeable boundary and impermeable boundary. The results of permeable boundary and impermeable boundary are depicted in Figs. 2 and 3, respectively. From these figures it is clear that Poisson ratio influences phase velocity. In general, the values of permeable boundary are greater than that of impermeable boundary. The difference between the values of permeable boundary and impermeable boundary increases as Poisson ratio increases. In the case of empty bone, phase velocity is

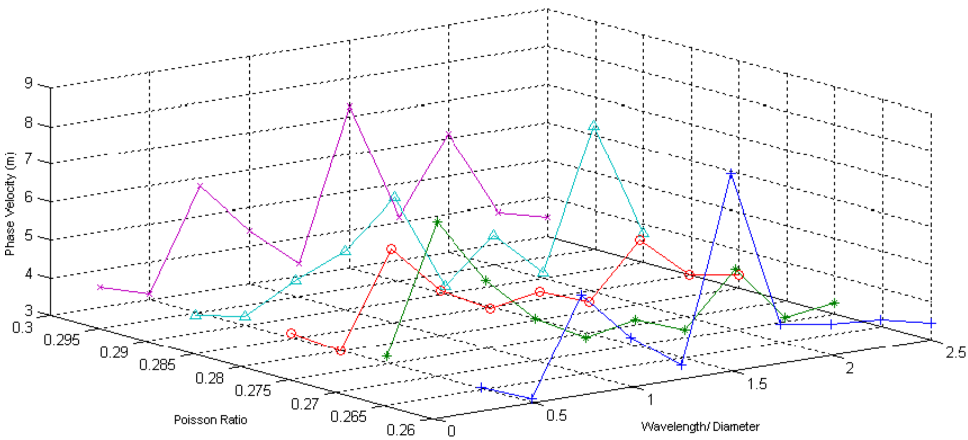

Fig. 3. Variation of phase velocity with wavelength/diameter at various poisson ratio in the case of empty bone for an impermeable boundary.

sensitive with respect to nature of the boundary and Poisson ratio. Using non-dimensional parameters into the frequency Eqs. (3.7) and (3.8), one obtains an implicit relation between phase velocity (ζ) and the ratio $\frac{\lambda}{D}$ for fixed $\frac{c_f}{V_3}$ and $\frac{\rho_f}{\rho}$. As in the case of empty bone, sensitivity analysis of phase velocity against Poisson ratio is made in the case of bone with marrow. For the numerical work, the velocity of sound in marrow (c_f) and the density of marrow ρ_f are taken to be $93\,\mathrm{m/s}$ and $900\,\mathrm{kg/m^3}$, respectively. These values are obtained in a private communication [10]. It is obtained that the phase velocity is not depending on Poisson ratio unlike the case of empty bone. Moreover, from the calculations it is seen that values for permeable boundary and impermeable boundary are equal. This means that nature

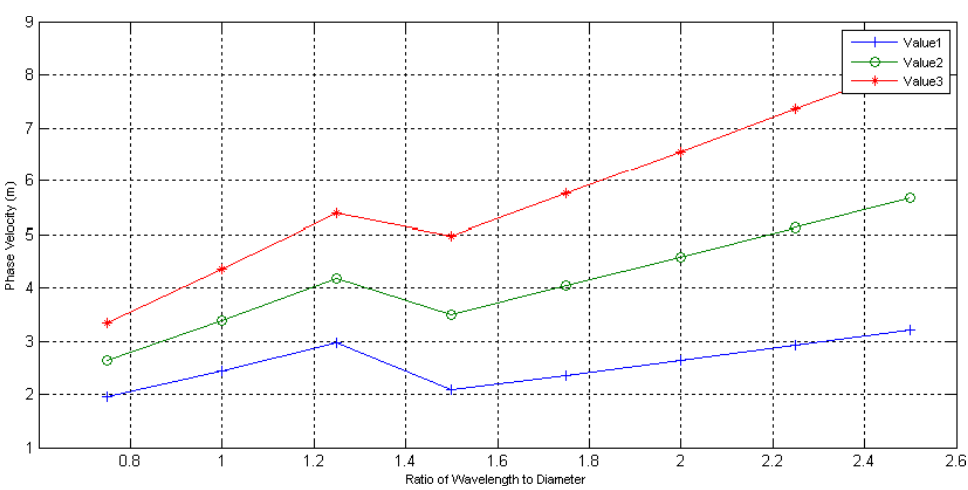

Fig. 4. Variation of phase velocity with wavelength/diameter in the case of bone with marrow.

of boundary does not have any influence over the values in the case of bone filled with marrow unlike empty bone. All these common results are depicted in Fig. 4. From Fig. 4, one can infer that first the values increase as the ratio increases up to $\frac{\lambda}{D} = 1.25$, next decreases up to 1.5 and then increases linearly after 1.5. From the figures, it is clear the values pertaining to empty bone are greater than that of bone filled with marrow. From the figures, it is evident that dispersive behaviors in the empty bone and the bone filled with marrow are different in phenomenon, in the sense, phase velocity is sensitive with respect to nature of boundary and Poisson ratio for empty bone, while it is not in the case of bone filled with marrow.

5. Conclusion

Cylindrical Bone filled with marrow is modelled as a cylindrical bore filled with fluid in the frame work of Biot's poroelastic theory to investigate the interaction at the interface of bone and marrow. A comparative dispersive study has been made between the empty bone and bone filled with marrow for both permeable and impermeable boundaries. The available data of bony materials has been used for the numerical evaluation. One limitation here is existing data is based on certain approximations. For more reliable data of bony materials, one has to make extensive experimental work. However, this Mathematical model and numerical evaluation give some rough estimate with regard to bone, in turn, this rough estimate may provide some deep insight into the mechanics of bone.

Acknowledgments

Thanks are due to anonymous referees for their suggestions that improved the presentation of this paper. This research is supported by Grant No.F.No.34-141\2008 (SR) from the University Grants Commission (UGC), Government of India.

References

[1] D. T. Felson, Epidemiology of hip and knee osteoarthritis, *Epidemiol Prev.* **10** (1988) 1–28.

[2] D. T. Felson, The incidence and natural history of knee osteoarthritis in the elderly, The farmingham osteoarthritis study, *Arthritis Rheum* **38** (1995) 1500–1505.

[3] A. A. Guccione, D. T. Felson, J. J. Anderson, J. M. Anthony, Y. Zhang, P. W. Wilson, M. Kelly-Hayes, P. A. Wolf, B. E. Kreger, W. B. Kamel, The effects of specific medical conditions on the functional limitations of elders in the Farmingham study, *Amer. J. Public Health* **84** (1994) 351–358.

[4] J. L. Nowinski and C. F. Davis, Propagation of longitudinal waves in circularly cylindrical bone elements, Transactions of ASME, *J. Appl. Mech.* (1971) 578–584.

[5] P. Malla Reddy and M. Tajuddin, Exact analysis of the plane strain vibrations of thick walled hollow poroelastic cylinders, *Int. J. Solids Struct.* **37** (2000) 3439–3456.

[6] M. A. Biot, The theory of propagation of elastic waves in fluid-saturated porous solid, *J. Acous. Soc. Amer.* **28** (1956) 168–178.

[7] P. Malla Reddy and M. Tajuddin, Wave propagation for cylindrical bore in poroelastic solid, in *Mathematics and Statistics in Engineering and Technology*, ed. A. Chattopadhyay (Narosa, 1999), pp. 47–51.
[8] P. Malla Reddy and M. Tajuddin, Cylindrical stress waves in poroelastic flat slabs, *J. Mech.* **22**(2) (2006) 161–165.
[9] M. A. Biot, Propagation of elastic waves in a cylindrical bore containing a fluid, *J. Appl. Phys.* **23**(9) (1952) 997–1005.
[10] R. R. Reddy, Orthopaedic Research Centre, New York University.